

COUNTING WEIGHTED INDEPENDENT SETS BEYOND THE PERMANENT*

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Abstract. Jerrum, Sinclair, and Vigoda [*J. ACM*, 51 (2004), pp. 671–697] showed that the permanent of any square matrix can be estimated in polynomial time. This computation can be viewed as approximating the partition function of edge-weighted matchings in a bipartite graph. Equivalently, this may be viewed as approximating the partition function of vertex-weighted independent sets in the line graph of a bipartite graph. Line graphs of bipartite graphs are perfect graphs and are known to be precisely the class of (claw, diamond, odd hole)-free graphs. So how far does the result of Jerrum, Sinclair, and Vigoda extend? We first show that it extends to (claw, odd hole)-free graphs, and then show that it extends to the even larger class of (fork, odd hole)-free graphs. Our techniques are based on graph decompositions, which have been the focus of much recent work in structural graph theory, and on structural results of Chvátal and Sbihi [*J. Combin. Theory Ser. B*, 44 (1988)], Maffray and Reed [*J. Combin. Theory Ser. B*, 75 (1999)], and Lozin and Milanić [*Discrete Algorithms*, 6 (2008), pp. 595–604].

Key words. independent set, counting, randomized algorithm, fully polynomial randomized approximation scheme, FPRAS, claw-free graph, fork-free graph, decomposition

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1. Introduction. Independent sets are central objects of study in graph theory.¹ In general, finding a largest independent set is a very hard problem. Indeed, it is known to be hard to approximate the size of this set within a ratio $n^{1-\varepsilon}$ for graphs on n vertices and any $\varepsilon > 0$, unless $P = NP$. (This follows from the equivalence to the maximum clique problem in the complementary graph and a result of Håstad [20]). This has led to an emphasis on studying this problem in particular classes of graphs.

In particular, there has been a focus on *hereditary* classes, that is, classes that are closed under vertex deletions. Equivalently, such a class can be defined by a (not necessarily finite) set of *forbidden (induced) subgraphs*.

Matchings are a particular case. A matching in a graph $G = (V, E)$ is an independent set in the *line graph* of the root graph G . (“Graph” will mean “simple undirected graph,” unless otherwise stated.) Edmonds [14] showed that a maximum weighted matching in any graph can be found in polynomial time. Beineke [1] showed that line graphs can be characterized by nine forbidden subgraphs. Of these, the *claw* (see Figure 1) seems the most important for algorithmic questions. Thus Minty [32] extended Edmonds’ algorithm to the larger class of *claw-free* graphs.

Claw-free graphs have been studied extensively by several authors, including Chudnovsky and Seymour in a long sequence of papers culminating in [9]. These

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¹Definitions not given here appear in subsection 1.1 below.

papers give a decomposition of claw-free graphs, which unfortunately is nonalgorithmic. However, this has been simplified and extended in [15, 33] to give an efficient decomposition which supports finding a maximum weighted independent set.

In this paper we are concerned with counting problems, and for these it is important to distinguish weighted and unweighted (or unary weighted) variants, even more so than with optimization problems. For example, there is an efficient approximation algorithm for counting unweighted matchings in a general graph, but the existence of an approximation algorithm for counting weighted matchings remains an open question. It is weighted counting problems that we focus on here. Hereditary classes are particularly suitable for counting problems, since they are self-reducible by vertex deletion.

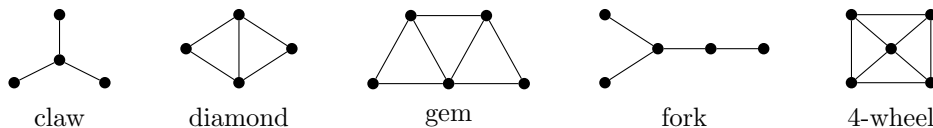
Claw-free graphs include line graphs. Therefore the $\#P$ -completeness result of Valiant [37] for matchings implies that *exact counting* of independent sets in polynomial time is unlikely. Even polynomial time *approximate* counting remains an open question for general line graphs in the weighted setting. However, building on an earlier pseudopolynomial algorithm of Jerrum and Sinclair [22], Jerrum, Sinclair, and Vigoda [23] made an important breakthrough in approximate counting. They gave a fully polynomial randomized approximation scheme (FPRAS) for counting weighted *perfect matchings* in a *bipartite* graph, the *permanent* approximation problem.

Our goal is to extend the result of [23] to larger classes of graphs. It might be expected that the right direction for this would be to matchings in general graphs but, as noted above, this remains an open problem, and a positive solution seems increasingly unlikely. An important requirement of the proof of [23] is that the graph should have no odd cycles, which places it precisely in the class of bipartite graphs. Indeed, Štefankovič, Vigoda, and Wilmes [39] have given a family of nonbipartite graphs for which the algorithm of [23] does not run in polynomial time. Interestingly, from the viewpoint of this paper, they also show that weighted matchings in these graphs, and in a more general class of graphs that are “close to bipartite,” can be counted in polynomial time, using the algorithm of [23] with a graph decomposition technique. In [39], this is the Gallai–Edmonds decomposition.

Here we take a different direction to generalize [23], regarding approximating the permanent as the problem of approximately counting weighted independent sets in line graphs of bipartite graphs. We show that these two problems are polynomial time equivalent. That approximating the permanent is reducible to approximately counting weighted independent sets is shown in subsection 2.5, and that counting arbitrarily weighted independent sets in line graphs of bipartite graphs is reducible to approximating the permanent is shown in subsection 2.4.4.

An important property of line graphs of bipartite graphs is that they are *perfect*. So it might be hoped that the appropriate generalization of the result of [23] would be to counting independent sets in perfect graphs. That this class can be recognized in polynomial time was shown by Chudnovsky et al. [5]. The maximum independent set in a perfect graph can also be found in polynomial time, using a convex optimization algorithm of Grötschel, Lovász, and Schrijver [17], though no combinatorial algorithm is yet known for this problem.

However, approximately counting independent sets in perfect graphs appears intractable in general. Bipartite graphs are perfect, but approximately counting independent sets in bipartite graphs defines the complexity class $\#BIS$. This class was introduced by Dyer et al. [13], and hardness for the class has since been used as evidence for the intractability of various approximate counting problems.

FIG. 1. *Claw, diamond, gem, fork and 4-wheel.*

Trotter [36] suggested the smaller class of *line-perfect* graphs. These are graphs whose line graph is perfect. Trotter showed that a graph is line-perfect if and only if it contains no odd cycle of size larger than three. Independent sets in the line graph (matchings in the root graph) appear to be a natural target for generalizing [23] but, in fact, they are a proper subclass of those that we will consider here.

Line graphs of bipartite graphs have a simple set of forbidden subgraphs [19]. These are the claw, all odd holes, and the *diamond* (see Figure 1). See Maffray and Reed [30, Thm. 4], who also gave the corresponding result [30, Thm. 5] for line graphs of bipartite *multigraphs*. These have the claw, *gem*, *4-wheel*, and odd holes (see Figure 1) as forbidden subgraphs. Of these, excluding the claw and the odd holes appears important in extending the algorithm of [23]. This results from the “canonical paths” argument used in its proof. However, the diamond, gem, or 4-wheel do not appear important in this respect.

Here we establish this claim. We extend the result of [23] to the class of graphs which excludes only claws and odd holes. This is essentially the class of *claw-free perfect* graphs. (See subsection 2.3 below.) These form a main focus of this paper, and we show that the algorithm of [23] can be extended to approximate the total weight of independent sets for graphs in this class. The structure of graphs in this class was characterized by Chvátal and Sbihi [10] and Maffray and Reed [30]. They gave a polynomial time *decomposition* algorithm that splits the graph into simpler parts. We use their results to show that the algorithm of [23] can be applied directly to count weighted independent sets in (claw, odd hole)-free graphs, a slightly larger class than claw-free perfect graphs. Since line graphs of bipartite graphs are a proper subclass of (claw, odd hole)-graphs, this is a natural generalization of the result of [23].

Chudnovsky and Plumettaz [6] have given a different decomposition of claw-free perfect graphs, which has the additional property of *composability*. That is, the rules used to decompose the graph can be applied in reverse to create precisely the graphs in the class (and no more). Unfortunately, this results in a considerably more complex decomposition than that of [10, 30]. We make no use of this here, since we do not need composability. Moreover, [6] does not give a polynomial time algorithm for its decomposition. The ideas in [15, 33] would give a polynomial time decomposition, but it is unclear whether this supports counting.

In section 2 we prove the following generalization of the result of [23].

THEOREM 1.1. *There is an FPRAS for counting all weighted independent sets in a (claw, odd hole)-free graph.*

Observe that the algorithm of [12] runs in polynomial time. It counts all weighted independent sets in an arbitrary claw-free graph $G = (V, E)$ with unary weights. (See also Matthews [31].) This generalizes Jerrum and Sinclair’s matching algorithm [22].

So what do we achieve by restricting to (claw, odd hole)-free graphs? The gain is that our algorithm is genuinely polynomial time, whereas that of [12] is only pseudopolynomial. In particular, this allows us to approximate the total weight of independent sets of any given size k . We can estimate the total weight of *maximum*

independent sets, which corresponds to counting maximum matchings in the root graph of a line graph. We then further relax the conditions on the class. We cannot relax first the odd hole condition, since this would take us into more general claw-free graphs, and might require counting matchings in general graphs. Therefore, we consider relaxing the claw-free condition.

Lozin and Milanić [28] described a polynomial time algorithm for finding a maximum weighted independent set in a *fork-free* graph. That is, a graph with only the *fork* (see Figure 1) as a forbidden subgraph. Clearly, this is a proper superclass of claw-free graphs, since the claw is a subgraph of the fork. In section 3, we show how our methods can be combined with the ideas of [28] to prove the following.

THEOREM 1.2. *There is an FPRAS for counting all weighted independent sets in a (fork, odd hole)-free graph.*

This gives a further nontrivial generalization of the result of [23].

1.1. Preliminaries. Let $\mathbb{N} = \{1, 2, \dots\}$ denote the natural numbers, and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. If $n \in \mathbb{N}$, let $[n] = \{1, 2, \dots, n\}$. For a set S , $S^{(2)}$ will denote the set of subsets of S of size exactly 2.

Throughout this paper, graphs are always simple and undirected. Let $G = (V, E)$ be a graph. We denote its vertex set by $V(G)$ and its edge set by $E(G) \subseteq V^{(2)}$. We write an edge $e \in E$ between v and w in G as $e = vw$, or $e = \{v, w\}$ if the vw notation is ambiguous. If $U, W \subseteq V$ with $U \cap W = \emptyset$, we will denote the U, W cut by $(U, W) = \{uw \in E : u \in U, w \in W\}$. We also consider *multigraphs*, in which E may have *parallel edges*. That is, E is a *multiset* with elements in $V^{(2)}$.

For a graph $G = (V, E)$, we will write $n = |V|$ and $m = |E|$, unless stated otherwise. The *empty graph* $G = (\emptyset, \emptyset)$ is the unique graph with $n = 0$. Also, $G = (V, V^{(2)})$, is the complete graph on n vertices. The *complement* of any graph $G = (V, E)$ is $\bar{G} = (V, V^{(2)} \setminus E)$.

The *neighborhood* of $v \in V$ will be denoted $N(v)$, and $N[v] = N(v) \cup \{v\}$. Then the *degree* $\deg(v)$ of v is $|N(v)|$. More generally, the neighborhood of a set $U \subseteq V$ is $N(U) = \{v \in V \setminus U : uv \in E \text{ for some } u \in U\}$, and $N[U] = U \cup N(U)$. We will say that a vertex $v \in V \setminus U$ is *complete* to U if $U \subseteq N(v)$ and *anticomplete* if $U \cap N(v) = \emptyset$. More generally, a set $U \subseteq V$ is complete to a set $W \subseteq V \setminus U$ if every $u \in U$ is adjacent to every $w \in W$ and anticomplete if $(U, W) = \emptyset$. Observe that this is a symmetric relation between U and W . The graph $G = (V, E)$ is *connected* if V cannot be partitioned into sets U, W that are anticomplete.

The term “induced” subgraph will always mean a vertex-induced subgraph. If $U \subseteq V$, we will write $G[U]$ for the subgraph of G induced by U . Where “subgraph” is used without qualification, it will always mean induced subgraph. Then a class \mathcal{C} of graphs is called *hereditary* if $G[U] \in \mathcal{C}$ for all $G \in \mathcal{C}$ and $U \subseteq V$. If $U \subseteq V$, we will often write $G \setminus U$ as shorthand for $G[V \setminus U]$.

We say a graph G *contains* a graph H if it has an induced subgraph isomorphic to H , and H is a *forbidden subgraph* for the graph class \mathcal{C} if no graph in \mathcal{C} contains H . It is easy to see that any hereditary class can be characterized by a (possibly infinite) set \mathcal{H} of minimal forbidden subgraphs. In this case we refer to \mathcal{C} as the class of \mathcal{H} -free graphs.

An *odd hole* in a graph G is a subset $H \subseteq V$, with $|H| \geq 5$ and odd, such that $G[H]$ is a simple cycle. A *perfect* graph G is such that neither G nor its complement \bar{G} contains an odd hole. A hole in \bar{G} is called an *antihole* in G . Perfect graphs were originally defined differently, but the equivalence to this definition was proved in [7].

The *line graph* $L(G) = (E, \mathcal{E})$ of a multigraph $G = (V, E)$ has $\mathcal{E} = \{\{xy, yz\} : xy, yz \in E\}$. We will write $G = L^{-1}(G')$ for the inverse operation, when it is defined, and call G the *root* multigraph of the line graph G' . Note that $G = L^{-1}(G')$ is not unique when G is a multigraph, whereas it is unique for $|V| > 4$ when G is a graph. When $L^{-1}(G')$ is not uniquely defined, we may choose it to be any multigraph G such that $G' = L(G)$.

A set $S \subseteq V$ is *independent* (or *stable*) in G if $G[S]$ is edgeless. The empty set \emptyset is an independent set in every graph. By $\mathcal{I}(G)$ we denote the set of all independent sets of G , and $\mathcal{I}_k(G) = \mathcal{I}(G) \cap V^{(k)}$ for $k \in \mathbb{N}_0$. The largest k for which $\mathcal{I}_k(G) \neq \emptyset$ is the *independence number* $\alpha(G)$ of G .

For further information on graph theory, see [4, 11], for example.

We will suppose that the vertices $v \in V(G)$ are equipped with nonnegative *weights* $w(v) \in \mathbb{R}$. We will denote such a vertex-weighted graph by (G, w) , or simply G when the vertex weights w are understood. Two weighted graphs $(G_1, w_1), (G_2, w_2)$ will be called *isomorphic* if G_1, G_2 are isomorphic as graphs, with no implied constraint on w_1 and w_2 . Since we are considering only approximation, we may assume here that $w(v) \in \mathbb{Q}$. The weight of a subset S of V is then defined to be $w(S) = \prod_{v \in S} w(v)$.² Then let $W_k(G) = \sum_{S \in \mathcal{I}_k(G)} w(S)$, and $W(G) = \sum_{S \in \mathcal{I}(G)} w(S) = \sum_{k=0}^{\alpha(G)} W_k(G)$. In particular, we have $W_0(G) = 1$, $W_1(G) = \sum_{v \in V} w(v)$, and $W_2(G) = \sum_{uv \notin E} w(u)w(v)$.

We will use only the following simple properties of $W(G)$. If G has connected components C_1, C_2, \dots, C_r , then $W(G) = \prod_{i=1}^r W(C_i)$, and if $\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_s$ partitions $\mathcal{I}(G)$, then $W(G) = \sum_{i=1}^s \sum_{I \in \mathcal{S}_i} w(I)$.

We will say that a vertex-weighted graph (G', w') is *equivalent* to a vertex-weighted graph (G, w) if $W_k(G) = W_k(G')$ for all $k \in \mathbb{N}_0$, and hence $W(G) = W(G')$. In particular, this implies $\alpha(G) = \alpha(G')$. Observe that equivalent weighted graphs are not necessarily isomorphic, and isomorphic weighted graphs are not necessarily equivalent.

Note that if $w(v) = 0$ for any $v \in V$, then G is equivalent to $G[V \setminus \{v\}]$, and thus we can consider such vertices as present in or deleted from G , whichever is more convenient. We will assume that such vertices are deleted before carrying out computations, so we may assume that $w(v) > 0$ for all $v \in V$.

If $w(v) = 1$ for all $v \in V$, then $W_k(G) = |\mathcal{I}_k(G)|$, the number of independent sets of size k in G , and $W(G) = |\mathcal{I}(G)|$ counts all independent sets in G . However, we also refer to the case with nonunit weights as “counting.”

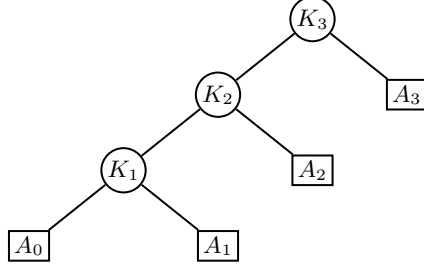
A central theme of structural graph theory has been *decomposition*, that is, breaking a graph into smaller pieces that have stronger properties than the original, such that the pieces are all connected to each other in some canonical fashion. Our counting algorithms for (claw, odd hole)-free and (fork, odd hole)-free graphs are based on two graph decompositions, clique cutset decomposition and modular decomposition, respectively. We will describe these in sections 2 and 3, respectively.

We consider approximating $W(G)$ and $W_k(G)$ in the following sense. An FPRAS (fully polynomial randomized approximation scheme) is an algorithm which produces an estimate \widehat{W} of a quantity W such that

$$\Pr((1 - \varepsilon)W \leq \widehat{W} \leq (1 + \varepsilon)W) \geq 3/4.$$

The key FPRAS we employ here is that of Jerrum, Sinclair, and Vigoda [23] for

²Note the difference from the corresponding definition $\sum_{v \in S} w(v)$ used in optimization.

FIG. 2. *Clique decomposition tree.*

the *permanent*. This uses the Markov chain approach to approximate counting, but we will not need the interior details of the algorithm. Essentially, we use [23] as a “black box” here. We note the equivalence of approximate counting with approximate random generation [24], but we make no direct use of this here.

For further information on approximate counting, see [21], for example.

2. Approximating $W(G)$ in (claw, odd hole)-free graphs. We develop an algorithm for approximating $W(G)$ in (claw, odd hole)-free graphs using clique cutset decomposition. But first we describe this method, and its application to counting, in a general setting.

2.1. Clique cutset decomposition. A *clique* K in a graph $G = (V, E)$ is a subset of V such that $G[K]$ is a complete graph. $K \subseteq V$ is a *clique cutset* of G if K is a clique and $G \setminus K$ is disconnected. In particular, \emptyset is a clique cutset of every disconnected graph. For a clique cutset K of G , let (A, B) be a partition of $V \setminus K$ such that A is anticomplete to B . The subgraphs $G[A \cup K]$ and $G[B \cup K]$ are *blocks* of the decomposition of G by K . These blocks may themselves have clique cutsets, so may contain further blocks. A block with no clique cutset is called an *atom*. The decomposition can be presented in the form of a tree, in which the interior vertices are cliques, and the leaves are atoms. Tarjan [34] gave an $O(mn)$ algorithm for a particular tree representation. This gives a binary decomposition tree in which all the interior nodes (cliques) form a path. If the tree has height h , we will number the atoms A_0, A_1, \dots, A_h and cliques K_1, K_2, \dots, K_h from the bottom up in this tree. While the atoms are all different, a clique can occur several times in a decomposition tree. See Figure 2, and see [34] for further information. We describe how this decomposition may be used for computing $W(G)$ in subsection 2.2.

2.2. Approximating $W(G)$ using clique cutset decomposition. Let \mathcal{C} be a hereditary class of graphs such that all graphs in \mathcal{C} have a clique cutset decomposition with all atoms in some hereditary class $\mathcal{A} \subset \mathcal{C}$, and we can approximate $W(G)$ (in the FPRAS sense) for any weighted $G = (V, E)$ in \mathcal{A} in time $T_{\mathcal{A}}(n) = \Omega(n)$, where T is assumed convex. We show how to approximate $W(G)$ for the entire graph G in time $T_{\mathcal{C}}(n) \leq 2nT_{\mathcal{A}}(n)$.

The decomposition tree in subsection 2.1 has cliques K_1, K_2, \dots, K_h and atoms A_0, A_1, \dots, A_h , where $h \leq n$. The root of the tree is K_h . Let $A'_i = A_i \setminus K_i$, $s_i = |K_i|$, $a'_i = |V(A'_i)|$. Let G_i be the graph formed by deleting the vertices of $A'_h, A'_{h-1}, \dots, A'_{i+1}$ from G . Thus $G_h = G$, and $G_{i-1} = G_i \setminus V(A'_i)$. We will account for the independent sets intersecting A'_i by revising the weights on the vertices in K_i , in a similar way to Tarjan’s [34] approach to the maximum weight independent set problem.

Since K_i is a clique cutset in G_i , we may partition $\mathcal{I}(G_i)$ by the value of $I \cap K_i$, which is either $\{v\}$ ($v \in K_i$) or \emptyset . Then we partition G_i into three vertex-disjoint subgraphs $G_{i-1} \setminus K_i$, $G[K_i]$, and A'_i . So we may write

$$\begin{aligned} W(G_i) &= W(G_{i-1} \setminus K_i) W(A'_i) + \sum_{v \in K_i} W(G_{i-1} \setminus N[v]) w(v) W(A'_i \setminus N(v)) \\ &= W(A'_i) \left(W(G_{i-1} \setminus K_i) + \sum_{v \in K_i} W(G_{i-1} \setminus N[v]) w(v) W(A'_i \setminus N(v)) / W(A'_i) \right) \\ &= W(A'_i) W(G_{i-1}), \end{aligned}$$

where the vertex weights in G_{i-1} relate to those in G_i by

$$w(v) \leftarrow w(v) W(A'_i \setminus N(v)) / W(A'_i) \quad (v \in K_i), \quad w(v) \leftarrow w(v) \quad (v \notin K_i).$$

Thus, since $G_h = G$, we may compute $W(G)$ by induction as

$$W(G) = W(A_0) \prod_{i=1}^h W(A'_i),$$

where we update the vertex weights as above at each stage. Note that the weights of some vertices may change several times in this process, since the cliques of Tarjan's decomposition are not necessarily vertex-disjoint.

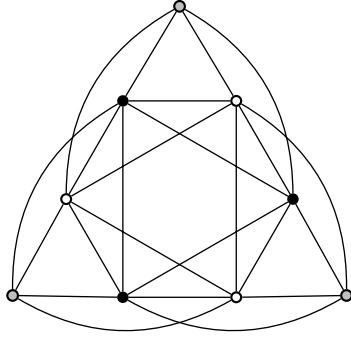
At stage i , we have to perform $s_i + 1$ computations on subgraphs of A'_i , which are all in \mathcal{A} . Thus the total time is $T_{\mathcal{C}}(n) = \sum_{i=1}^h (s_i + 1) T_{\mathcal{A}}(a'_i) \leq 2n T_{\mathcal{A}}(n)$, since $h \leq n$, $s_i \leq n$, $\sum_{i=1}^h a'_i \leq n$ and T is convex.

Note that this analysis deals only with applications of the algorithm for \mathcal{A} . It ignores the effect of the bit-size of the vertex weights on $T_{\mathcal{A}}(n)$. This distinction is not so important for optimization, but is much more important for counting, since contracting modules (see subsection 3.1) can cause exponential growth in the weights. This may cause the running time estimate to increase by a small polynomial factor. The same comment applies to the algorithm of section 3. However, we do not pursue this issue further in this paper.

2.2.1. Error analysis. We are only approximating the weight of graphs in \mathcal{A} , so we must show that the resulting error in $W(G)$ can be controlled for $G \in \mathcal{C}$.

Suppose we approximate to a factor $(1 \pm \varepsilon/n^2)$ throughout. Then, by induction, the weights in A_{h-i} will have relative error at most $(1 \pm \varepsilon/n^2)^i$. Thus the estimate of the total weight of A_0 will have relative error at most $(1 \pm \varepsilon/n^2)^h$. The error in $W(A'_{h-i})$ will be at most $(1 \pm \varepsilon/n^2)^i$, so the error in $\prod_{i=1}^h W(A'_i)$ is at most $(1 \pm \varepsilon/n^2)^{h(h-1)/2}$. Hence the error in $W(G)$ is at most $(1 \pm \varepsilon/n^2)^{h(h+1)/2}$. Since $h < n-1$, the error is at most $(1 \pm \varepsilon/n^2)^{n^2/2}$, which is at most $(1 \pm \varepsilon)$ for $\varepsilon < 1$. So the overall error can be kept within any desired relative error ε by performing the weight estimations for all graphs in \mathcal{A} to within error ε/n^2 .

2.3. Structure of claw-free perfect graphs. In our application of the method of subsection 2.2, \mathcal{C} will be the class of (claw, odd hole)-free graphs, and \mathcal{A} will be a class of graphs that we will define below. We must examine approximate counting in this class, but first we review the structural results which allow us to apply clique cutset decomposition.

FIG. 3. *Minimal peculiar graph from [10].*

Chvátal and Sbihi [10] investigated the structure of claw-free *perfect* graphs as a special class of perfect graphs. These are closely related to (claw, odd hole)-free graphs. The difference is that odd antiholes are also forbidden. The following lemma of Rebea explains that relation.

LEMMA 2.1 (Rebea; see [10]). *Let G be a connected claw-free graph with $\alpha(G) \geq 3$. If G contains an odd antihole, then it contains a hole of length five.*

COROLLARY 2.2. *A claw-free graph with $\alpha(G) \geq 3$ is perfect if and only if it has no odd hole.*

Proof. From the strong perfect graph theorem [7], Lemma 2.1 implies that a claw-free graph with $\alpha(G) \geq 3$ is perfect if and only if it contains no odd hole. \square

Chvátal and Sbihi [10] gave a decomposition theorem via clique cutsets for claw-free perfect graphs. As described in subsection 2.1, a clique cutset decomposition can be described by a binary tree whose interior vertices are cliques and whose leaves are atoms.

THEOREM 2.3 (Chvátal and Sbihi). *If a claw-free perfect graph has no clique cutset, then it is either elementary or peculiar.*

We will describe elementary and peculiar graphs below. These will be the atoms of the decomposition.

2.3.1. Peculiar graphs. A *peculiar* graph is constructed as follows. A set K of vertices, initially a clique, is partitioned into six nonempty subsets $A_1, A_2, A_3, B_1, B_2, B_3$. At least one edge is from each of the edge sets (A_1, B_2) , (A_2, B_3) , and (A_3, B_1) . Finally, three disjoint nonempty cliques K_1, K_2, K_3 are added, and each vertex in K_i is made adjacent to every vertex in $K \setminus (A_i \cup B_i)$ for $i = 1, 2, 3$.

The smallest peculiar graph with $|A_i|, |B_i|, |K_i| = 1$ ($i = 1, 2, 3$) is shown in Figure 3. The black vertices are A 's, the white B 's, and the grey K 's. This graph is a template for all peculiar graphs, as shown by Chvátal and Sbihi [10].

We will need the following simple observation about peculiar graphs.

LEMMA 2.4. *A peculiar graph $G = (V, E)$ has independence number $\alpha(G) = 3$. Any independent set of size three has one vertex in each of K_1, K_2, K_3 .*

Proof. Note that $K_1 \cup A_3 \cup B_2$, $K_2 \cup A_1 \cup B_3$, $K_3 \cup A_2 \cup B_1$ are three cliques which cover V , so $\alpha(G) \leq 3$. However, we can form an independent set of size three by taking one vertex from each of K_1, K_2, K_3 , so $\alpha(G) \geq 3$.

Let $I = \{v_1, v_2, v_3\}$ be any maximum independent set in G . Suppose first $I \cap K_i = \emptyset$ for all $i = 1, 2, 3$. Then I is contained in K . But K is a perfect graph with vertices contained in two disjoint cliques, $A_1 \cup A_2 \cup A_3$ and $B_1 \cup B_2 \cup B_3$. Thus K is a cobipartite graph, with $\alpha(K) \leq 2$. Hence $\alpha(G) \leq 2$, a contradiction.

Thus, without loss of generality, assume $v_1 \in K_1 \cap I$. Now $N[v] = (K \cup K_1) \setminus (A_1 \cup B_1)$. So $v_2, v_3 \in A_1 \cup B_1 \cup K_2 \cup K_3$. But $A_1 \cup B_1 \cup K_i$ is a clique for $i = 2, 3$. Thus, if $v_2 \in A_1 \cup B_1$, v_3 cannot exist. Thus $v_2 \in K_2$, without loss of generality, and then we must have $v_3 \in K_3$. \square

In Figure 3, the three corner triangles cover all the vertices, and the three corner vertices form an independent set.

Peculiar graphs do not form an hereditary class. If in a subgraph G of a peculiar graph, $K_i = \emptyset$ holds for any $i = 1, 2, 3$, it follows that $\alpha(G) \leq 2$. However, if $\alpha(G) \leq 2$, then G is a clique or a cobipartite graph. But both of these are elementary graphs, as defined in subsection 2.3.2 below. Thus we may insist that a peculiar graph has $K_i \neq \emptyset$ ($i = 1, 2, 3$) and $\alpha(G) = 3$.

2.3.2. Elementary graphs. Chvátal and Sbihi called a graph $G = (V, E)$ *elementary* if E can be two-(edge)-colored so that edges $xy, yz \in E$ have distinct colors whenever $xz \notin E$. Such a coloring is called elementary. It is clear from this that elementary graphs form a hereditary class. Whether G has an elementary coloring can be checked by forming the *Gallai graph* $\text{Gal}(G) = (E, \mathcal{E})$, where $\{xy, yz\} \in \mathcal{E}$ if and only if (x, y, z) is a P_3 . Clearly $\text{Gal}(G)$ can be constructed in time $O(mn)$, by taking all pairs of $xy \in E$ and $z \in V$, and checking that $yz \in E$, $xz \notin E$. Then an elementary coloring exists if and only if $\text{Gal}(G)$ is bipartite. If so, the two color classes of $\text{Gal}(G)$ give an elementary coloring of G . This can be recognized in $O(|\mathcal{E}|)$ time by breadth-first search, so this gives an $O(mn)$ algorithm for recognizing elementary graphs and determining an elementary coloring. Note that $\text{Gal}(G)$ may be disconnected. For example, if G is a triangle, $\text{Gal}(G)$ is an independent set of size three. In this case, we color each component separately, assigning the two color classes arbitrarily.

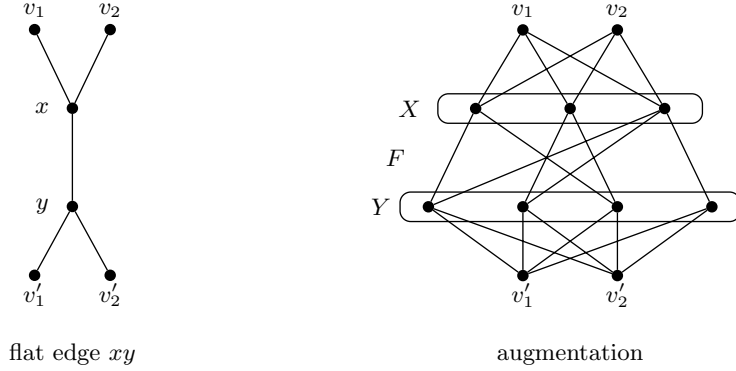
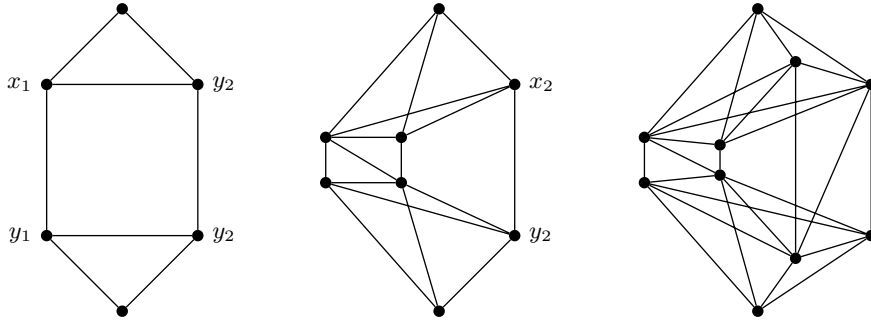
Maffray and Reed [30] characterized elementary graphs in a very precise way. They showed the following.

THEOREM 2.5 (Maffray and Reed). *G is elementary if and only if it is an augmentation of the line graph of bipartite multigraph.*

We must describe the “augmentation” in this theorem. An edge xy in G is called *flat* if x, y have no common neighbor. Then we augment the flat edge by replacing x by a clique X , y by a clique Y , and xy by any nonempty edge set $F \subseteq (X, Y)$. That is, we replace xy by a cobipartite graph called an *augment*. Finally, we add all edges between X and $N(x) \setminus \{y\}$ and all edges between Y and $N(y) \setminus \{x\}$ (see Figure 4). Then a graph G' is an augmentation of a graph G if G' can be obtained from G by applying one or more such steps to independent flat edges in G . See Figure 5, where the first graph is a line graph of a bipartite multigraph, as we will show below. The second and third show augmentations using the *independent* flat edges x_1y_1 and then x_2y_2 and two 2×2 cobipartite graphs as replacements.

We observe that an augmentation of a line graph of a bipartite multigraph need not be a line graph of a bipartite multigraph, as can be seen in Figure 5, where the two augmentations contain a gem which is an excluded structure for the class of line graphs of bipartite multigraphs by the following characterization of this class.

THEOREM 2.6 (Maffray and Reed [30]). *A graph is the line graph of bipartite multigraph if and only if it is (claw, gem, 4-wheel, odd hole)-free. (See Figure 1.)*

FIG. 4. *Augmenting a flat edge.*FIG. 5. *Augmentation of flat edges.*

Maffray and Reed [30] show how to recover the structure of an elementary graph as the line graph of a bipartite multigraph with augmented flat edges, using an elementary coloring of the graph. This can be done in $O(mn)$ time, so there is an $O(mn)$ time algorithm for determining the graph structure. (Maffray and Reed claim only the looser bound $O(m^2)$.)

2.4. Counting in (claw, odd hole)-free graphs. We conclude the following from Corollary 2.2, Lemma 2.4, and Theorem 2.3.

LEMMA 2.7. *Every (claw, odd hole)-free graph G without a clique cutset and with $\alpha(G) > 3$ is elementary.*

This gives us a clique cutset decomposition in which the atoms are in the hereditary class \mathcal{A} of graphs that are either elementary or have $\alpha(G) \leq 3$. To apply the method of subsection 2.2, we must consider how to approximate $W(G)$ in these graphs.

2.4.1. Computing $W(G)$ in graphs with $\alpha(G) \leq 3$. Let G be a (claw, odd hole)-free atom. For any k , we can determine $\mathcal{I}_k(G)$ in $O(n^k)$ time by listing all k -tuples of vertices and checking which are independent in G . Thus we can determine $W_k(G)$ for $k = 0, 1, 2, 3, 4$ in $O(n^4)$ time. If $W_4(G) > 0$, we conclude that G must be elementary, by Lemma 2.7. Otherwise, we set $W(G) = \sum_{k=0}^3 W_k(G)$.

2.4.2. Approximating $W(G)$ in elementary graphs. If a (claw, odd hole)-free atom G has $\alpha(G) > 3$, then it is elementary by Lemma 2.7. We use the $O(mn)$ time algorithm of Maffray and Reed [30] to identify G as the line graph of a bipartite

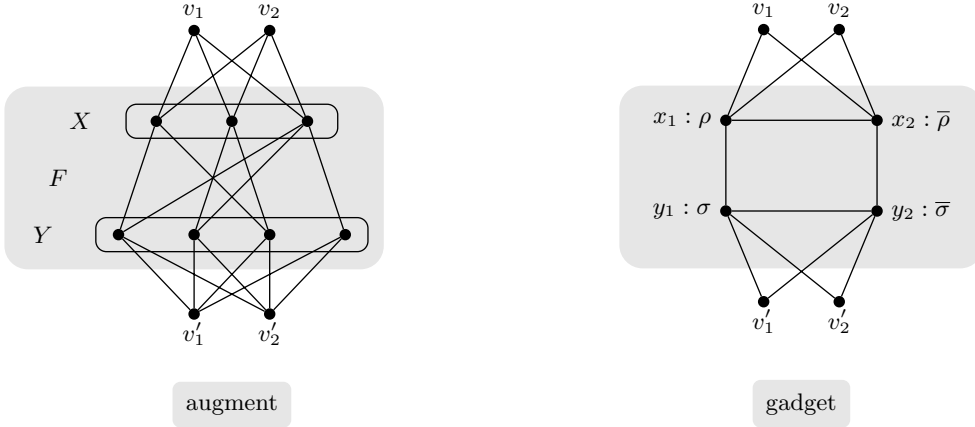


FIG. 6. Augmentation and equivalent gadget with vertex weights.

multigraph with augments. If this algorithm fails, we conclude that the original graph was not (claw, odd hole)-free, and halt. Otherwise, we have an elementary atom, and we continue.

An elementary graph G is not necessarily the line graph of a bipartite multigraph because of the augments, as discussed in subsection 2.3.2. However, we will replace G by an equivalent G' , such that G' is the line graph of a bipartite multigraph. We do this by replacing the augments in G by “gadgets” which are line graphs of bipartite multigraphs.

2.4.3. Augmentation gadgets. Suppose the augment $Z = X \cup Y$ in $G = (V, E)$, with vertex weights $w(v)$ ($v \in V$), comprises a cobipartite graph with cliques on X, Y and connecting bipartite graph $(X \cup Y, F)$. Clearly $W_k(Z) \neq 0$, only for $k = 0, 1, 2$.

If $U \subseteq V$, we will write $W_k(U)$ for $W_k(G[U])$. Then $W_1(X) = \sum_{v \in X} w(v)$, $W_1(Y) = \sum_{v \in Y} w(v)$, and $W_2(Z) = \sum_{uv \notin F} w(u)w(v)$. These are, respectively, the total weights of independent sets in Z which involve X alone, Y alone, or both. We will also write $\bar{W}_2(Z) = W_1(X)W_1(Y) - W_2(Z) = \sum_{uv \in F} w(u)w(v)$.

Consider the gadget Z' shown in Figure 6, where $\rho, \sigma, \bar{\rho}, \bar{\sigma} \geq 0$ are to be determined. Let $X' = \{x_1, x_2\}$, $Y' = \{y_1, y_2\}$. Note that both vertices in the clique X' have the same neighbors external to Z' as all vertices in X have to vertices external to Z , and similarly for Y' and Y .

If we set $\bar{\rho} = W_1(X) - \rho$, then $W_1(X') = \rho + \bar{\rho} = W_1(X)$, and if we set $\bar{\sigma} = W_1(Y) - \sigma$, then $W_1(Y') = \sigma + \bar{\sigma} = W_1(Y)$. Thus the total weight of independent sets using X' but not Y' is $W_1(X)$, and the total weight of independent sets using Y' but not X' is $W_1(Y)$, as required.

The weight of independent sets using both X' and Y' is

$$W_2(Z') = \rho\bar{\sigma} + \bar{\rho}\sigma = \rho(W_1(Y) - \sigma) + \sigma(W_1(X) - \rho) = W_2(Z),$$

again as required, provided

$$\rho = \frac{W_2(Z) - \sigma W_1(X)}{W_1(Y) - 2\sigma}.$$

It is convenient to break symmetry by requiring $\sigma < \bar{\sigma}$, making the denominator of the above fraction positive. We must also have $\rho, \bar{\rho} \geq 0$, so $0 \leq \rho \leq W_1(X)$. Thus we

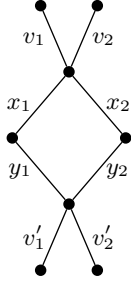


FIG. 7. Root graph of the gadget depicted in Figure 6.

require

$$0 \leq \sigma \leq \min\{W_2(Z), \overline{W}_2(Z)\}/W_1(X).$$

Otherwise we can choose σ arbitrarily. Then the gadget Z' is equivalent to Z . The particular choice $\sigma = 0$ deletes y_1 and its incident edges and gives an even smaller gadget. Note that Z' is itself an augment, equivalent to Z for computing $W_k(G)$ for any $0 \leq k \leq \alpha(G)$.

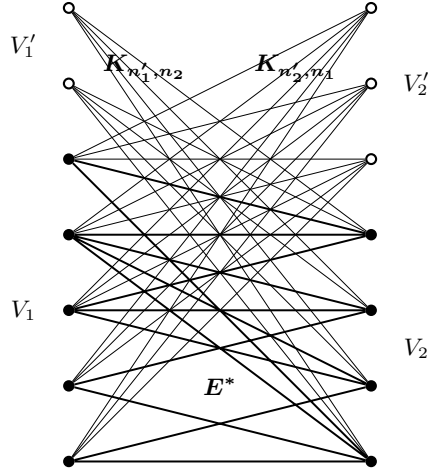
The gadget Z' is the line graph of a simple bipartite graph (see Figure 7). Every augment Z that has fewer vertices than Z' is also line graph of a suitable bipartite graph. So using this gadget to replace all augments on at least four vertices in G will result in it becoming the line graph G' of bipartite multigraph, as required. Moreover, the size of G' does not exceed the size of G , which becomes relevant in the following analysis.

2.4.4. Reduction to permanent approximation. We now need to determine $W_k(G')$, where G' is the line graph of a bipartite multigraph. We determine its root multigraph $G'' = L^{-1}(G')$ and verify that it is bipartite in $O(m)$ time, using (for example) the algorithm of Lehot [26]. The vertex weights in the line graph G' become edge weights in G'' , and independent sets become matchings. The graph G'' may have parallel edges, but for our purposes we can reduce this to a simple edge-weighted bipartite graph G^* by adding the weights on parallel edges. Now $M_k(G^*) = W_k(G')$ will be the total weight of all *matchings* of size k in G^* .

Suppose $G^* = (V_1 \cup V_2, E^*)$, where $n_1 = |V_1|$, $n_2 = |V_2|$. We wish to use the permanent algorithm of [23] to determine $M_k(G^*)$. However, this algorithm only computes $M_n(G^*)$ for the perfect matching case $n_1 = n_2 = n$. It is possible the algorithm of [23] can be modified to the general case, but the general case can be reduced to the permanent, as follows.

To determine $M_k(G^*)$, let $n'_1 = n_2 - k$, $n'_2 = n_1 - k$. We add a set V'_1 of n'_1 vertices to G^* , and the edges of a complete bipartite graph $K_{n'_1, n_2} = (V'_1 \cup V_2, V'_1 \times V_2)$, and add a set V'_2 of n'_2 vertices and the edges of a complete bipartite graph $K_{n_1, n'_2} = (V_1 \cup V'_2, V_1 \times V'_2)$. See Figure 8, where $n_1 = 5$, $n_2 = 4$, $k = 2$. The weights assigned to the added edges are all 1. Let this weighted graph be $G^+ = (V_1^+ \cup V_2^+, E^+)$, where $V_1^+ = V_1 \cup V'_1$, $V_2^+ = V_2 \cup V'_2$, and E^+ is E^* plus the edges of the complete bipartite graphs. Let $n^+ = |V_1^+| = |V_2^+| = n_1 + n_2 - k = n - k$.

Now observe that there is a correspondence between matchings of size k in G^* and perfect matchings in G^+ . For each k -matching in G^* , we match the $n_1 - k$ unmatched vertices in V_1 with vertices in V'_2 , and we match the $n_2 - k$ unmatched vertices in V_2 with vertices in V'_1 . Given a perfect matching M^+ in G^+ , we can uniquely recover a k -

FIG. 8. *Equivalent permanent problem.*

matching M^* in G^* of the same weight. However, there are $n'_1!n'_2! = (n_1 - k)!(n_2 - k)!$ matchings M^+ corresponding to any M^* . Thus $M_{n^+}(G^+) = (n_1 - k)!(n_2 - k)!M_k(G^*)$.

Thus our algorithm will use the permanent method of [23] to compute $M_{n^+}(G^+)$, and then divide this by $(n_1 - k)!(n_2 - k)!$ to obtain $M_k(G^*)$. Thus we can determine $W_k(G')$ for any elementary graph G' and any $0 \leq k \leq \alpha(G')$. We then compute $W(G') = \sum_{k=1}^{\alpha(G')} W_k(G')$. This has a time complexity of $O(nT_P(n, \varepsilon))$, where $T_P(n, \varepsilon)$ is the time to approximate the total weight of perfect matchings in an $n \times n$ bipartite graph with relative error $1 \pm \varepsilon$. The clique decomposition in subsection 2.2 gives another n factor, so the overall time complexity of approximating $W(G)$ in a (claw, odd hole)-free graph G with n vertices is $T_C(n, \varepsilon) = O(n^2T_P(n, \varepsilon n^{-2}))$, where the accuracy parameter comes from the error analysis in subsection 2.2.1. The best bound for $T_P(n, \varepsilon)$ known is $O(n^7 \log^4 n + n^6 \log^5(n) \varepsilon^{-2})$ [2]. Thus the overall time complexity for (claw, odd hole)-free graphs is $T_C(n, \varepsilon) = O(n^{12} \log^5(n) \varepsilon^{-2})$. This analysis is clearly loose and could be tightened. However, without a radical improvement in the bound for $T_P(n, \varepsilon)$, the overall time complexity cannot be improved to anything practically relevant.

2.5. Approximating $W_k(G)$. The freedom to use very large vertex weights allows us to approximate $W_k(G)$ for any $0 \leq k \leq \alpha(G)$. For $k < \alpha(G)$, we would need to use the algorithm described in [12], which is an improvement of the approach of that in [22]. This method is based on the fact that the *independence polynomial*

$$(2.1) \quad P_G(\lambda) = \sum_{i=0}^{\alpha(G)} \lambda^i W_i(G)$$

has only real negative roots when G is claw-free. This was proved in [8] for unit weights and extended to general weights in [12]. However, the algorithm requires the reduction from approximate counting to approximate random generation [24]. Therefore, we will not give further details here, but it is a straightforward application of the above algorithm for $W(G)$.

Since we can compute $W_k(G)$ in line graphs of multigraphs or peculiar graphs, it might be expected that we could do this directly through the decomposition. In

fact, we compute $W(G)$ for line graphs of multigraphs by computing all the $W_k(G)$. However, there are two obstacles to approximating $W_k(G)$ efficiently in this way. First, the gadgets may replace a large number of vertices by at most four vertices, so it is difficult to keep track of k through the permanent computations. Second, the process used in subsection 2.2, replacing subgraphs by revised weights on vertices, makes it difficult to track $W_k(G)$ efficiently by combining contributions from different modified vertices. polynomially bounded weights, this is possible by a dynamic programming approach, but this case is resolved in [12] for all claw-free graphs. However, for weights exponential in n , there can be an exponential number of cases, which must be combined efficiently somehow. The same problem arises in our use of modular decomposition in subsection 3.1 below.

However, for $\alpha = \alpha(G)$ the total weight $W_\alpha(G)$ of maximum independent sets, which correspond exactly to perfect matchings in a graph when G is a line graph, there is a simpler approach, which we will describe. Note that this gives a complete generalization of the result of [23].

We have a multiplier λ for every vertex weight, as in (2.1) above, so $w(v) \leftarrow \lambda w(v)$ for all $v \in V$. Then $W(G)$ becomes

$$W(\lambda G) = \sum_{I \in \mathcal{I}(G)} w(I) \lambda^{|I|} = \sum_{k=0}^{\alpha(G)} W_k(G) \lambda^k = P_G(\lambda).$$

Thus, if $\lambda \geq 1$ and $\alpha = \alpha(G)$,

$$W_\alpha(G) \lambda^\alpha \leq W(\lambda G) < W(G) \lambda^{\alpha-1} + W_\alpha(G) \lambda^\alpha$$

so

$$W_\alpha(G) \leq W(\lambda G) / \lambda^\alpha < W_\alpha(G) + W(G) / \lambda.$$

So we need $\lambda \geq W(G) / \varepsilon W_\alpha(G)$ to achieve relative error ε . Let $w_{\min} \leq w(v) \leq w_{\max}$ for all $v \in V$. Then $W(G) \leq 2^n w_{\max}^\alpha$ and $W_\alpha(G) \geq w_{\min}^\alpha$, since G has at most 2^n independent sets and at least one of size α . Thus it suffices to take $\lambda \geq 2^n (w_{\max} / w_{\min})^\alpha / \varepsilon$ in order that $W(\lambda G) / \lambda^\alpha$ approximates $W_\alpha(G)$ with relative error ε . Since the time complexity of the algorithm is polynomial in $\log \lambda$, it is clearly polynomial in n and the bit size of the $w(v)$'s, as required. Here we have suppressed the approximation in computing the W s, though this may increase the relative approximation in a similar way to 2.2.1 above and 3.6 below.

3. Approximating $W(G)$ in (fork, odd hole)-free graphs. We will extend the result for claw-free graphs to fork-free graphs using modular decomposition, as described in subsection 3.1. Our algorithm is inspired by Lozin and Milanič's [28] approach to computing the maximum weight independent set problem using modular decomposition. Again, we will first describe the modular decomposition approach in a general context.

3.1. Modular decomposition. *Modules* were first introduced by Gallai [16], [29, Thm. 3.1.2], using different terminology. If $S \subseteq V$ in $G = (V, E)$, we say that any vertex $x \in V \setminus S$ *distinguishes* S if there exist $u, v \in S$ with $ux \in E$, $vx \notin E$. Then a set $M \subseteq V$ is a *module* of G if no vertex of $V \setminus M$ distinguishes it. Alternatively, M is a module if $N(u) \setminus M = N(v) \setminus M$ holds for all $u, v \in M$. Thus \emptyset , V , and all the singletons $\{v\}$ ($v \in V$) are modules of G . These are the *trivial* modules; all other modules are *nontrivial*.

Another way of defining a module M is that every vertex $v \in V \setminus M$ must be either complete or anticomplete to M . It follows that M is a module in G if and only if it is a module in \overline{G} , since v is complete to M in G if and only if it is anticomplete to M in \overline{G} . It is also easy to show that the modules of G are closed under intersection. However, if M_1, M_2 are modules, then $M_1 \cup M_2$ is only guaranteed to be a module if $M_1 \cap M_2 \neq \emptyset$. Otherwise, some v could be complete to M_1 and anticomplete to M_2 , so neither complete nor anticomplete to $M_1 \cup M_2$. Modules are also not generally closed under complementation, since if $u \notin M$ is complete to M and $v \notin M$ is anticomplete to M , then any vertex of M distinguishes $V \setminus M$.

OBSERVATION 3.1. *If M is a module of $G = (V, E)$ and $U \subseteq V$, then $M \cap U$ is a module of $G[U]$.*

Proof. Otherwise, two vertices $u, v \in M \cap U$ distinguishable by $x \in U \setminus M$ belong to M , that is, $u, v \in M$, and are distinguished by $x \in V \setminus M$. \square

If $M \neq \emptyset$ is a module of G , then G/M denotes the graph obtained from G by contracting M to a single vertex, with the same adjacencies in G/M as all vertices in M . We will label this vertex as v_M in G/M . For $|M| \leq 1$ let $G/M = G$. Note the following.

OBSERVATION 3.2. *If M is a module in $G \in \mathcal{C}$ for some hereditary class \mathcal{C} , then $G/M \in \mathcal{C}$.*

Proof. Contracting M deletes all but one of its vertices and relabels the remaining vertex v_M . Since \mathcal{C} is a class of unlabelled graphs, $G/M \in \mathcal{C}$ by heredity. \square

If M is a module of G , and M' is a module of $G[M]$, then M' is also a module of G .

A module that does not overlap with any other module is strong [25]; more formally, $M \subseteq V$ is a *strong module* of $G = (V, E)$ if M is a module of G and for all modules M' of G we have $M \subseteq M'$ or $M \supseteq M'$ or $M \cap M' = \emptyset$. Every trivial module of G is strong. G is a *prime graph* if every strong module of G is trivial. Like connectedness, primeness is an intrinsic property of the graph, but not a hereditary property. For example, none of the graphs in Figure 1 is prime.

The strong modules of $G = (V, E)$ are partially ordered by set inclusion. The unique top element of this poset is V , and \emptyset is the unique bottom element. The layer above \emptyset consists of the singletons $\{v\}$ for all $v \in V$. The strong modules in the next layer up are called *prime modules* of G . A prime module M of G induces a prime subgraph $G[M]$.

Let M_1 and M_2 be strong modules such that $M_1 \subset M_2$. We say M_2 *covers* M_1 if, for all strong modules M , $M_1 \subseteq M$ and $M \subseteq M_2$ imply $M_1 = M$ or $M = M_2$. If $|V| > 1$, then the strong modules of $G = (V, E)$ except those of size at most one are the nodes of a tree rooted at V , where the arcs of the tree are given by the cover relation. Note that the singleton modules are often included as leaves in this tree, but we will not do so here. See Figure 9, where $M_7 = V$. We will call this the *standard decomposition tree*. Equivalent definitions exist (see [18]). Several algorithms are known to compute the standard decomposition tree in linear time (see, for example, [35]).

We will use this tree in an equivalent form. Thereto we number the modules $M_1, M_2, \dots, M_h = V$, according to a *postorder* on the standard tree. This order places all the descendants of a vertex before the vertex itself, as in Figure 9. (See, for example, [38, Chap. 3].) We will call this the *extended decomposition tree*. Here $G_0 = G$, and $G_i = G_{i-1}/\bar{M}_i$ ($i \in [h]$), where \bar{M}_i is M_i after M_1, \dots, M_{i-1} have been contracted

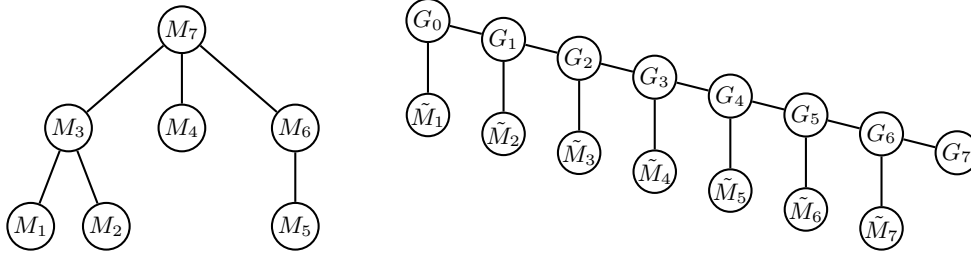


FIG. 9. A standard modular decomposition tree and its extended tree.

in order to single vertices. We will denote this by $G_{i-1} = G/(M_1, M_2, \dots, M_{i-1})$, and similarly and hence $\tilde{M}_i = M_i/(M_1, M_2, \dots, M_{i-1})$ is a module in G_{i-1} . We can represent the extended decomposition tree as shown in Figure 9, where G_{i-1} ($i \in [h]$) are the internal vertices, and $\tilde{M}_1, \tilde{M}_2, \dots, \tilde{M}_h$ and G_h are the leaves.

Note that \tilde{M}_i is a prime module in G_{i-1} , and in particular $G_{i-1}[\tilde{M}_i]$ is a prime graph. Also G_h is a single vertex, since $M_h = V$ has been contracted to a single vertex. Note also that M_i is isomorphic to the graph obtained by contracting only its children in the standard tree, but its vertex weights require contracting the whole subtree of which it is the root.

Let $t_i = |\tilde{M}_i|$, so $2 \leq t_i < n$. Then $|V(G_0)| = n$, and $|V(G_i)| = |V(G_{i-1})| - t_i + 1$, so $1 = |V(G_h)| = n - \sum_{i=1}^h t_i + h \leq n - 2h + h = n - h$. Thus $h \leq n - 1$, so the decomposition tree contains at most $(n-1)$ modules. Also $n - \sum_{i=1}^h t_i + h = 1$ implies $\sum_{i=1}^h t_i = n - h - 1 \leq n - 2$. Thus the extended decomposition tree can be represented explicitly in $O(n)$ space.

We can compute the extended tree from the standard tree in a further $O(n)$ time, using postorder tree traversal [38]. We can contract modules and form the modules \tilde{M}_i , during the traversal. Since $\sum_{i=1}^h |\tilde{M}_i| < n$, the additional time complexity for contracting modules is also $O(n)$, so the total remains $O(m)$. Of course, this excludes the time to compute the vertex weight of v_{M_i} in G_i , as will be detailed in subsection 3.2 below. However, these computations can also be integrated into the tree traversal. Performing the algorithm this way, the extended tree is purely a useful notional device and never computed explicitly.

Of course, we could compute the extended tree explicitly by successively finding a prime module and contracting it, until the contracted graph is prime. While this may be conceptually simpler, it is computationally inefficient. Finding any prime module appears to be an $\Omega(m)$ computation, so the time complexity of producing the whole extended tree becomes $\Omega(mn)$. The inefficiency clearly results from discarding information gained in earlier searches when carrying out the later searches.

We will make use of the following.

OBSERVATION 3.3. *Each of the modules \tilde{M}_i ($i \in [h]$) in the extended tree is isomorphic to a prime subgraph of G .*

Proof. Since \tilde{M}_i is M_i with all its submodules contracted, from Observation 3.2 follows that \tilde{M}_i is isomorphic to some subgraph \tilde{M}'_i of G . Note that \tilde{M}'_i is not unique. As observed above, each of the \tilde{M}_i is prime, so \tilde{M}'_i is also prime. Since they are vertex-weighted graphs, the isomorphism between \tilde{M}_i and \tilde{M}'_i is in the sense defined in subsection 1.1. \square

3.2. Approximating $W(G)$ using modular decomposition. Let \mathcal{C} be a hereditary class, and $\mathcal{P} \subseteq \mathcal{C}$, the (nonhereditary) class of prime graphs in \mathcal{C} . Let $T_{\mathcal{C}}(n)$ be the time to compute $W(G)$ for any connected n -vertex graph $G \in \mathcal{C}$, and let $T_{\mathcal{P}}(n)$ bound the time to compute $W(G)$ for any n -vertex prime graph $G \in \mathcal{C}$. We may assume that $T_{\mathcal{P}}(n)$ is a monotonically increasing function that is linear or convex. We will show $T_{\mathcal{C}}(n) \leq T_{\mathcal{P}}(n) + O(m)$. This strengthens the result of [28, Thm. 1], with an easier proof.

We use the notation of subsection 3.1. We construct the extended decomposition tree as described in subsection 3.1, and begin with $G_0 = G$. At step i , we contract the module \tilde{M}_i in G_{i-1} to give G_i , giving $v_{\tilde{M}_i}$, the vertex that represents \tilde{M}_i in G_i , weight $W(\tilde{M}_i)$. Then $W(G_i) = W(G_{i-1})$, since the set \tilde{M}_i has the same neighborhood in G_{i-1} as the vertex M_i in G_i . Thus, by induction, $W(G) = W(G_0) = W(G_h) = w(v)$ for the unique vertex $v \in V(G_h)$.

If $h = 1$, $T_{\mathcal{C}}(n) = T_{\mathcal{P}}(n)$. Otherwise, $2 \leq h \leq n - 1$ so, omitting the time to compute the modular decomposition, we have

$$\begin{aligned} T_{\mathcal{C}}(n) &\leq \max \left\{ \sum_{i=1}^h T_{\mathcal{P}}(t_i) : \sum_{i=1}^h t_i = n + h, 2 \leq t_i \leq n, i \in [h] \right\} \\ &\leq T_{\mathcal{P}}(n - h + 2) + (h - 1)T_{\mathcal{P}}(2), \\ &\leq T_{\mathcal{P}}(n) + (n - 2)T_{\mathcal{P}}(2) = T_{\mathcal{P}}(n) + O(n), \end{aligned}$$

where the second line follows from the first since $T_{\mathcal{P}}$ is convex, so $\sum_{i=1}^h T_{\mathcal{P}}(t_i)$ is maximized by setting $t_1 = n - h + 2$, $t_i = 2$, $i = 2, \dots, h$. The third line follows from the second because $T_{\mathcal{P}}$ is increasing, so $T_{\mathcal{P}}(n - h + 2) \leq T_{\mathcal{P}}(n)$ for $h \geq 2$. Adding the $O(m)$ time to compute the modular decomposition [35], we have $T_{\mathcal{C}}(n) \leq T_{\mathcal{P}}(n) + O(m)$. Thus we can approximate $W(G)$ in any graph in \mathcal{C} , with only an $O(m)$ overhead, if we can approximate it in all the prime graphs in \mathcal{C} .

Note that this analysis deals only with applications of the algorithm for \mathcal{P} , as does that in [28, Thm. 1]. It ignores the effect of the bit-size of the vertex weights on $T_{\mathcal{P}}(n)$. This distinction is not so important for optimization, but is much more important for counting, since contracting modules can cause exponential growth in the weights. However, we will not pursue this issue further.

3.3. Structure of fork-free graphs. Lozin and Milanić [28] used a modular decomposition approach to determine the maximum weight independent set in a fork-free graph. However, there seems to be a flaw in their algorithm and its analysis. Consequently, we will rework most of their development, in addition to extending it from optimization to counting.

The approach of [28] is based on a structural result given in [28, Thm. 3]. We begin with a more useful version of this theorem, the original being too weak for its application. We first repeat two structural lemmas from [28]. We will also make use of the following simple observation, which was used as the basis of the algorithm in [35].

OBSERVATION 3.4. *If v is any vertex of G , and M is a module not containing v , then either $M \subseteq N(v)$ or $M \subseteq V \setminus N[v]$.*

Proof. Otherwise, v distinguishes M , contradicting it being a module. \square

LEMMA 3.5 (see [28, Thm. 3] and also [3]). *If a prime fork-free graph contains a claw, then it contains one of the graphs H_1, \dots, H_5 (see Figure 10).*

LEMMA 3.6 (see [28, Lemma 1]). *Let G be a fork-free graph and let v be any vertex of G . Assume that $H \in \{H_1, \dots, H_5\}$ is an induced subgraph of $G \setminus N[v]$. Then no neighbor of v distinguishes $V(H)$.*

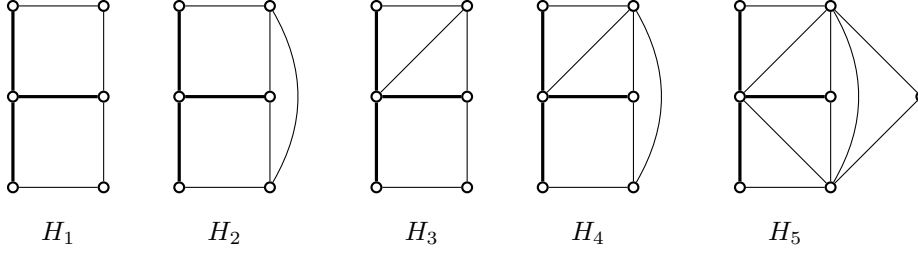


FIG. 10. The five minimal fork-free prime graphs extending a claw.

Note that we could omit H_2 for our application, because it contains a 5-hole, but we give the result for general claw-free graphs, as used in [28], since what follows could be used for computing a maximum weight independent set in a claw-free graph.

The statement and proof of the following theorem modifies the weaker result of [28, Thm. 3]. An earlier conference version [27, Thm. 4.1] also provides this.

THEOREM 3.7. *Let G be a fork-free graph and v a vertex of G . If G is prime and M is a prime subgraph of $G \setminus N[v]$, then M is claw-free.*

Proof. Assume by contradiction that M contains a claw. Then by Lemma 3.5, M contains H , one of the graphs H_1, \dots, H_5 . Hence, by Lemma 3.6, $N(v)$ can be partitioned into sets Y and Z , such that Y is anticomplete to H and Z is complete to H .

Let W be an (inclusionwise) maximal subset of vertices of $G \setminus N[v]$ satisfying the following properties:

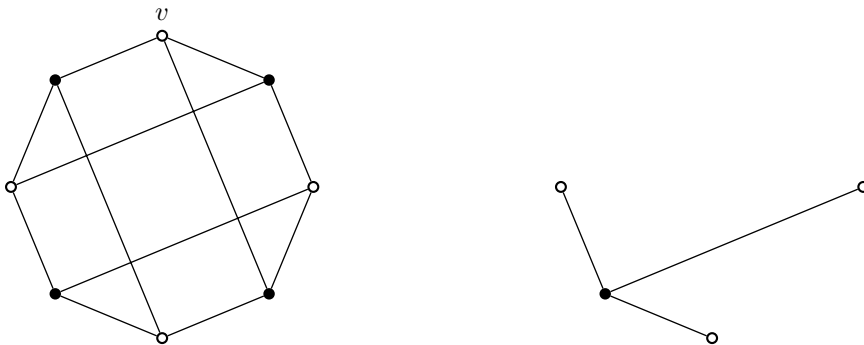
- (i) $V(H) \subseteq W$,
- (ii) $G[W]$ is connected,
- (iii) $\overline{G[W]}$ is connected,
- (iv) Z is complete to W ,
- (v) Y is anticomplete to W .

Note that such a set W exists since $V(H)$ satisfies all these properties. Clearly, $5 < |V(H)| \leq |W| < |V(G)|$. Since G is prime, W cannot be a nontrivial module of G , and hence some $u \in V(G) \setminus W$ distinguishes W . Note that $u \neq v$. We will obtain a contradiction (to the existence of W) by showing that the set $W' = W \cup \{u\}$ also satisfies properties (i)–(v).

Since W satisfies (iv) and (v), $u \in V \setminus N(v)$ and hence $W' \subseteq V \setminus N(v)$. Clearly W' satisfies (i). Since W satisfies (ii) and (iii), and since u has both a neighbor and a nonneighbor in W , it follows that W' satisfies (ii) and (iii).

Suppose that u has a nonneighbor $z \in Z$. Since u distinguishes W and $\overline{G[W]}$ is connected, u distinguishes a pair of nonadjacent vertices $w_1, w_2 \in W$. But then $\{u, w_1, z, w_2, v\}$ induces a fork, a contradiction. Therefore, Z is complete to W' and hence W' satisfies (iv).

Finally, suppose that u has a neighbor $y \in Y$. Since $G[W]$ is connected, and u distinguishes W , there is a shortest path $P = (v_0, \dots, v_k)$ connecting $V(H)$ and u in $G[W']$ with $v_0 \in V(H)$ and $v_k = u$. Let $v_{k+1} = y$ and $v_{k+2} = v$. Note that v is anticomplete to $V(P)$, and y is anticomplete to $V(P) \setminus \{v_k\}$, and hence (v_0, \dots, v_{k+2}) is a chordless path. Since v_2 has no neighbor in H , by Lemma 3.6, v_1 is complete to $V(H)$. But then any two nonadjacent vertices of H , together with v_1, v_2, v_3 , induce a fork in G , a contradiction. Therefore, Y is complete to W' and hence W' satisfies (v). \square

FIG. 11. Graph G and a derived $G \setminus N[v]$.

Note that Theorem 3.7 cannot obviously be strengthened. If G is a fork-free prime graph and v a vertex of G , then $G \setminus N[v]$ is not necessarily claw-free nor prime. Consider, for example, an $n \times n$ complete bipartite graph with a perfect matching removed. This is easily shown to be fork-free and prime. The case $n = 4$ is shown in Figure 11. The graph $G \setminus N[v]$ for the vertex labelled v is also shown. This graph is clearly neither claw-free nor prime. Consequently, Theorem 3 of [28] is inapplicable, even in this simple case.

3.4. Approximating $W(G \setminus N[v])$ for prime G and $v \in V$. To apply Theorem 3.7, we need the following strengthening.

COROLLARY 3.8. *Let G be prime and v be a vertex of G . The modules \tilde{M}_i ($i \in [h]$) in the extended decomposition tree for $G \setminus N[v]$ are claw-free.*

Proof. This follows directly from Observation 3.3 and Theorem 3.7. \square

To determine $W(G)$ for a prime graph, we first show how to determine $W(G \setminus N[v])$ for any $v \in V$. Let G_v denote $G \setminus N[v]$. As we have seen, G_v is not prime in general, so we must approximate $W(G_v)$ using the modular decomposition approach of subsection 3.1.

The algorithm is then as follows. We construct the extended decomposition tree for G_v with modules \tilde{M}_i ($i \in [h]$). For each $i = 1, 2, \dots, h$, we determine $W(G_{i-1}[\tilde{M}_i])$, using this as the weight for $v_{\tilde{M}_i}$ in G_i . From Corollary 3.8, $G_{i-1}[\tilde{M}_i]$ is claw-free, so we may use the algorithm of section 2 in this computation. Finally $W(G_v) = w(u)$, where u is the unique vertex in G_h .

More generally, suppose $G \in \mathcal{C}$ for some hereditary class \mathcal{C} and, given $v \in V$, all prime subgraphs of G_v are in some smaller hereditary class \mathcal{A} . Then we can use this method to approximate $W(G_v)$ for graphs in \mathcal{C} , using modular decomposition and an algorithm for approximating W for graphs in \mathcal{A} . In our application $\mathcal{C} = (\text{fork, odd hole})$ -free and $\mathcal{A} = (\text{claw, odd hole})$ -free.

3.5. Approximating $W(G)$ for prime G . The algorithm described in subsection 3.4 approximates $W(G \setminus N[v])$ for prime G and $v \in V$. Let $S(v) = \{I \in \mathcal{I}(G) : v \in I\}$. Then $w(v)W(G \setminus N[v]) = \sum_{I \in S(v)} w(I)$, the total weight of all independent sets containing v . The classes \mathcal{C}, \mathcal{A} are as in subsection 3.4.

We can write $V = \{v_1, v_2, \dots, v_n\}$ and determine $w(v_i)W(G \setminus N[v_i])$ for $i \in [n]$, similarly to [28], but the sum of these greatly overestimates $W(G)$, since $\{S(v_i) : i \in [n]\}$ is a cover of $\mathcal{I}(G)$, not a partition.

Let $V_i = \{v_i, \dots, v_n\}$ and $S'(v_i) = \{I \in \mathcal{I}(G) : v_i \in I \text{ and } I \subseteq V_i\}$. The sets $\{S'(v_i) : i \in [n]\}$ form a partition of $\mathcal{I}(G) \setminus \{\emptyset\}$, and so

$$(3.1) \quad W(G) = 1 + \sum_{i=1}^n \sum_{I \in S'(v_i)} w(I) = 1 + \sum_{i=1}^n w(v_i) W(G[V_i \setminus N[v_i]]).$$

So we must approximate $W(G[V_i \setminus N[v_i]])$ for $i \in [n]$. We do this by constructing the extended decomposition tree for $G \setminus N[v_i]$ with leaf modules $\tilde{M}_1, \tilde{M}_2, \dots, \tilde{M}_h$, as in subsection 3.4. From Corollary 3.8 we know that the modules $\tilde{M}_1, \tilde{M}_2, \dots, \tilde{M}_h$ in this decomposition are in \mathcal{A} . We transform this extended decomposition tree of $G \setminus N[v_i]$ into an extended decomposition tree for $G[V_i \setminus N[v_i]]$. For a fixed $i \in [n]$ we take $G'_0 = G[V_i \setminus N[v_i]]$ and for $j \in [h]$ we set $G'_j = G_j \setminus \{v_1, v_2, \dots, v_{i-1}\}$ and $\tilde{M}'_j = \tilde{M}_j \setminus \{v_1, v_2, \dots, v_{i-1}\}$. For each $j \in [h]$ the set \tilde{M}'_j is a module in G'_{j-1} by Observation 3.1.

To compute $W(G[V_i \setminus N[v_i]])$, we note that restricting to $V_i \setminus N[v_i]$ is equivalent to putting $w(v) = 0$ for all $v \notin V_i \setminus N[v_i]$. Thus we can use the algorithm of subsection 3.4, with exactly the same justification, after setting $w(v) = 0$ for $v \notin V_i \setminus N[v_i]$. Of course, in carrying out the algorithm we actually delete the vertices in $V \setminus (V_i \setminus N[v_i])$. Thus the algorithm approximates $W(G[V_i \setminus N[v_i]])$ for $i \in [n]$, using the algorithm of section 2, and then combines the estimates using (3.1).

3.6. Approximating $W(G)$ for all graphs in $G \in \mathcal{C}$. Since we can now approximate $W(G)$ for any prime G , we can use the algorithm of subsection 3.2 to lift this to arbitrary G . This completes the description of our algorithm.

The algorithm will fail if the (claw, odd hole)-free algorithm fails on any of the modules \tilde{M}'_j for any prime G and any $G[V_i \setminus N[v_i]]$ ($i \in [h]$). In that case we conclude that G is not (fork, odd hole)-free and terminate.

The basis of the algorithm is modular decomposition. From subsection 3.2 this gives only a negligible overhead to the algorithm for prime graphs in \mathcal{C} . From subsection 3.4, the algorithm for $G \setminus N[v]$ is modular decomposition, so this adds a negligible overhead. However, the algorithm for prime G in subsection 3.5 requires n applications of the algorithm for $G \setminus N[v]$. Thus if $T_{\mathcal{A}}(n, \varepsilon)$ is the time complexity of the subroutine for \mathcal{A} , the overall time complexity is $O(nT_{\mathcal{A}}(n, \varepsilon n^{-2}))$. (The error analysis is similar to the one seen earlier in the reduction to permanent approximation and is given below.) For $\mathcal{A} = (\text{claw, odd hole})$ -free graphs, $T_{\mathcal{A}}(n, \varepsilon) = O(n^{12} \log^5(n) \varepsilon^{-2})$, so for $\mathcal{C} = (\text{fork, odd hole})$ -free graphs, $T_{\mathcal{C}}(n, \varepsilon) = O(n^{17} \log^5(n) \varepsilon^{-2})$.

3.6.1. Error analysis. We can only approximate the total weight for graphs in \mathcal{A} , in our application the class of (claw, odd hole)-free graphs. So we must show that the resulting error in $W(G)$ can be controlled for all $G \in \mathcal{C}$, in our application the class of (fork, odd hole)-free graphs.

Suppose we approximate to a factor $(1 \pm \varepsilon/2n^2)$ for $G \in \mathcal{A}$. In step i of the algorithm of subsection 3.2, we approximate $W(\tilde{M}_i)$ and contract \tilde{M}_i with this weight. Thus one weight in G_{i+1} has error $(1 \pm \varepsilon/2n^2)$. We do this at most n times, so the error in $W(G)$ becomes at most $(1 \pm \varepsilon/2n)$. We do this n times in the method of subsection 3.5 and add the estimates. However, this does not increase the relative error. Finally, we apply the algorithm of subsection 3.2 again, so the error is at most $(1 \pm \varepsilon/2n)^n$, which is at most $(1 \pm \varepsilon)$ for $\varepsilon < 1$.

3.7. Approximating $W_{\alpha}(G)$. We cannot use the method of subsection 2.5 to approximate $W_k(G)$ for arbitrary $0 \leq k \leq \alpha(G)$, because there is no known analogue for fork-free graphs of the real-rootedness result of [9] for claw-free graphs.

Note also that we cannot use the decomposition directly to do this, for the reason discussed in subsection 2.5. We could use the method of [12] to approximate $W_k(G)$ in the leaf modules of the decomposition tree, but then we may have the same exponential complexity for combining these estimates as we process the tree. This approach seems possible for polynomial-sized weights using the dynamic programming, as suggested in subsection 2.5. This would give an extension to the results in [12], but that is outside the scope of this paper.

However, the method given in subsection 2.5 for estimating $W_\alpha(G)$ is valid for any graph class, not necessarily hereditary, where we can use arbitrary vertex weights. Therefore, the result of [23] for approximating the permanent can be completely generalized to approximating the total weight of maximum independent sets in (fork, odd hole)-free graphs.

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